where m_0 is the object's mass at rest, v is its speed, and c is the speed of light. What is this speed limit? (We explore this problem further in **Example 2.12**.)

The idea of a limit is central to all of calculus. We begin this chapter by examining why limits are so important. Then, we go on to describe how to find the limit of a function at a given point. Not all functions have limits at all points, and we discuss what this means and how we can tell if a function does or does not have a limit at a particular value. This chapter has been created in an informal, intuitive fashion, but this is not always enough if we need to prove a mathematical statement involving limits. The last section of this chapter presents the more precise definition of a limit and shows how to prove whether a function has a limit.

2.1 | A Preview of Calculus

Learning Objectives

2.1.1 Describe the tangent problem and how it led to the idea of a derivative.

2.1.2 Explain how the idea of a limit is involved in solving the tangent problem.

2.1.3 Recognize a tangent to a curve at a point as the limit of secant lines.

2.1.4 Identify instantaneous velocity as the limit of average velocity over a small time interval.

2.1.5 Describe the area problem and how it was solved by the integral.

2.1.6 Explain how the idea of a limit is involved in solving the area problem.

2.1.7 Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

The Tangent Problem and Differential Calculus

Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f(x) = -2x - 3$, $g(x) = \frac{1}{2}x + 1$, and $h(x) = 2$, shown in **Figure 2.2**.
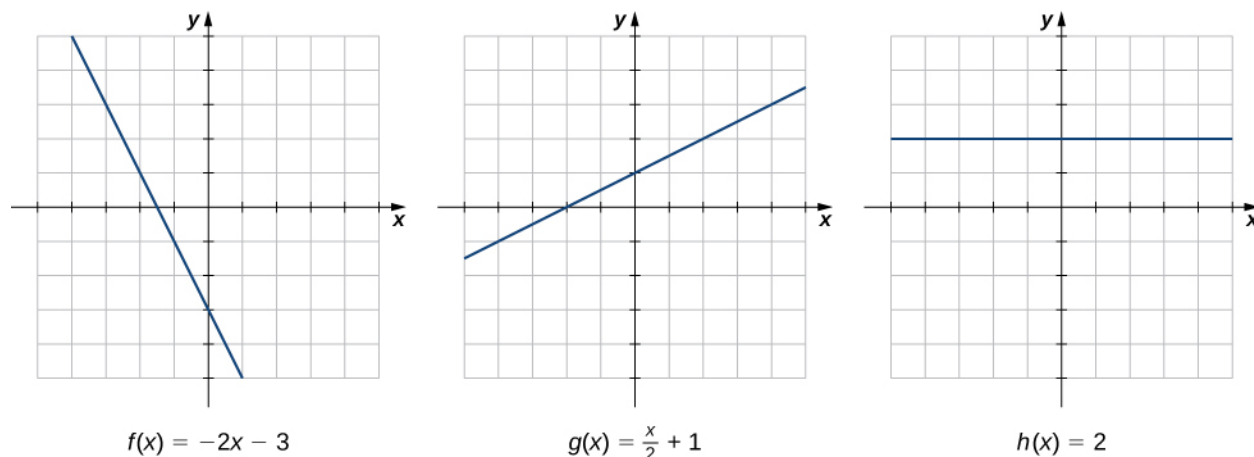

$$f(x) = -2x - 3 \qquad g(x) = \frac{x}{2} + 1 \qquad h(x) = 2$$

Figure 2.2 The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

As we move from left to right along the graph of $f(x) = -2x - 3$, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the x-axis, the y-coordinate decreases by 2 units. This rate of change is determined by the slope (-2) of the line. Similarly, the slope of $1/2$ in the function $g(x)$ tells us that for every change in x of 1 unit there is a corresponding change in y of $1/2$ unit. The function $h(x) = 2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k(x) = x^2$ (**Figure 2.3**). The graph of $k(x) = x^2$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?
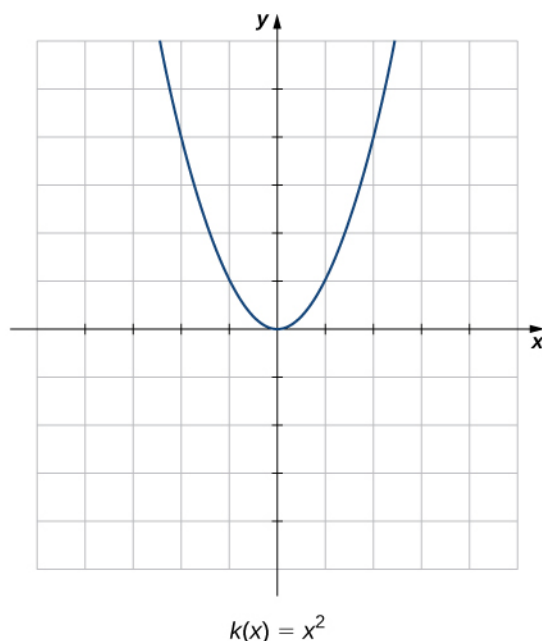

$$k(x) = x^2$$

Figure 2.3 The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a **secant** line. **Figure 2.4** shows a secant line to a function $f(x)$ at a point $(a, f(a))$.

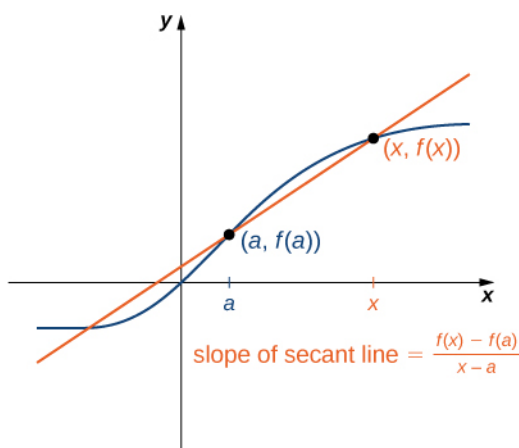slope of secant line $= \dfrac{f(x) - f(a)}{x - a}$

Figure 2.4 The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Definition

The **secant** to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by

$$m_{\sec} = \frac{f(x) - f(a)}{x - a}.$$

(2.1)

The accuracy of approximating the rate of change of the function with a secant line depends on how close x is to a. As we see in **Figure 2.5**, if x is closer to a, the slope of the secant line is a better measure of the rate of change of $f(x)$ at a.
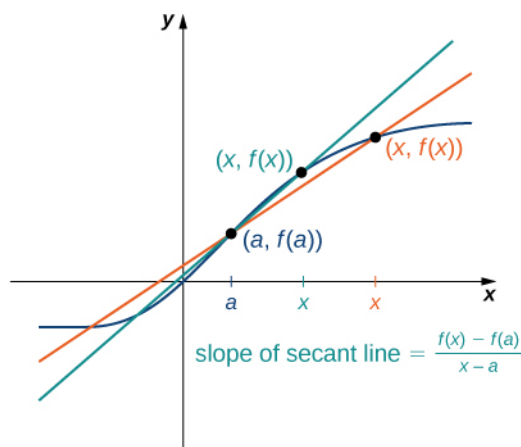

Figure 2.5 As x gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at a (**Figure 2.6**). The slope of the tangent line to the graph at a measures the rate of change of the function at a. This value also represents the derivative of the function $f(x)$ at a, or the rate of change of the function at a. This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

For an interactive demonstration of the slope of a secant line that you can manipulate yourself, visit this applet (*Note:* this site requires a Java browser plugin): **Math Insight (http://www.openstaxcollege.org/l/ 20_mathinsight)** .
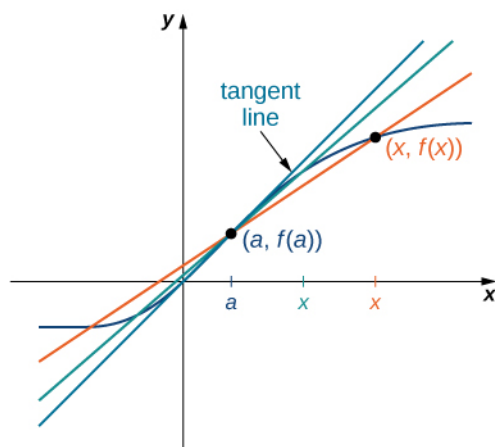

Figure 2.6 Solving the Tangent Problem: As x approaches a, the secant lines approach the tangent line.

Example 2.1 illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

Example 2.1

Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and each of the following points on the graph of $f(x) = x^2$.

a. $(2, 4)$

b. $\left(\frac{3}{2}, \frac{9}{4}\right)$

Solution

Use the formula for the slope of a secant line from the definition.

a. $m_{\sec} = \frac{4 - 1}{2 - 1} = 3$

b. $m_{\sec} = \frac{\frac{9}{4} - 1}{\frac{3}{2} - 1} = \frac{5}{2} = 2.5$

The point in part b. is closer to the point $(1, 1)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 (**Figure 2.7**).
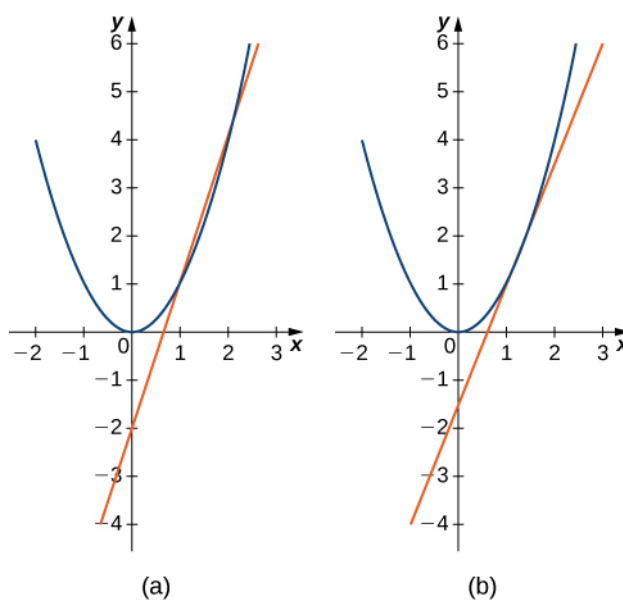

Figure 2.7 The secant lines to $f(x) = x^2$ at $(1, 1)$ through (a) $(2, 4)$ and (b) $\left(\frac{3}{2}, \frac{9}{4}\right)$ provide successively closer approximations to the tangent line to $f(x) = x^2$ at $(1, 1)$.

2.1 Estimate the slope of the tangent line (rate of change) to $f(x) = x^2$ at $x = 1$ by finding slopes of secant lines through $(1, 1)$ and the point $\left(\frac{5}{4}, \frac{25}{16}\right)$ on the graph of $f(x) = x^2$.

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s(t)$, that gives the position of an object along a coordinate axis at any given time t. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t = a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the **average velocity** of an object over a time period to be the change in its position divided by the length of the time period.

Definition

Let $s(t)$ be the position of an object moving along a coordinate axis at time t. The **average velocity** of the object over a time interval $[a, t]$ where $a < t$ (or $[t, a]$ if $t < a$) is

$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}.$$ **(2.2)**

As t is chosen closer to a, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point a, we let the x-values approach a in the slope of the secant line. Similarly, to find the instantaneous velocity at time a, we let the t-values approach a in the average velocity. This process of letting x or t approach a in an expression is called taking a **limit**. Thus, we may define the **instantaneous velocity** as follows.

Definition

For a position function $s(t)$, the **instantaneous velocity** at a time $t = a$ is the value that the average velocities approach on intervals of the form $[a, t]$ and $[t, a]$ as the values of t become closer to a, provided such a value exists.

Example 2.2 illustrates this concept of limits and average velocity.

Example 2.2

Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground t seconds later (for $0 \leq t \leq 2$) is given by $s(t) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

    a. $[0.49, 0.5]$

    b. $[0.5, 0.51]$

Solution

Substitute the data into the formula for the definition of average velocity.

    a. $v_{\text{ave}} = \dfrac{s(0.49) - s(0.5)}{0.49 - 0.5} = -15.84$

b. $v_{\text{ave}} = \dfrac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.016$

The instantaneous velocity is somewhere between −15.84 and −16.16 ft/sec. A good guess might be −16 ft/sec.

2.2 An object moves along a coordinate axis so that its position at time t is given by $s(t) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may be interpreted as the area under a curve. This leads us to ask the question: How can we find the area between the graph of a function and the *x*-axis over an interval (**Figure 2.8**)?
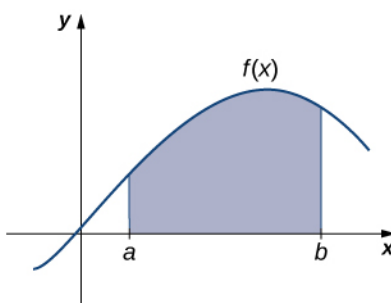

Figure 2.8 The Area Problem: How do we find the area of the shaded region?

As in the answer to our previous questions on velocity, we first try to approximate the solution. We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up the areas of these rectangles (**Figure 2.9**).
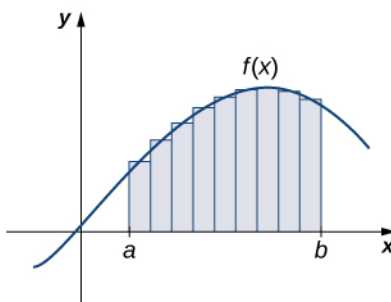

Figure 2.9 The area of the region under the curve is approximated by summing the areas of thin rectangles.

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the *x*-axis over the interval $[a, b]$. Once again, we find ourselves taking a limit. Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

Example 2.3

Estimation Using Rectangles

Estimate the area between the *x*-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in **Figure 2.10**.
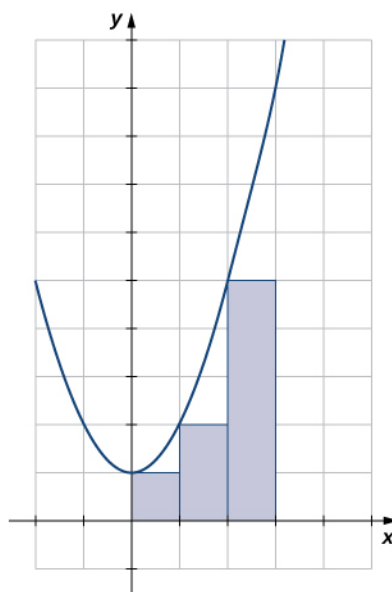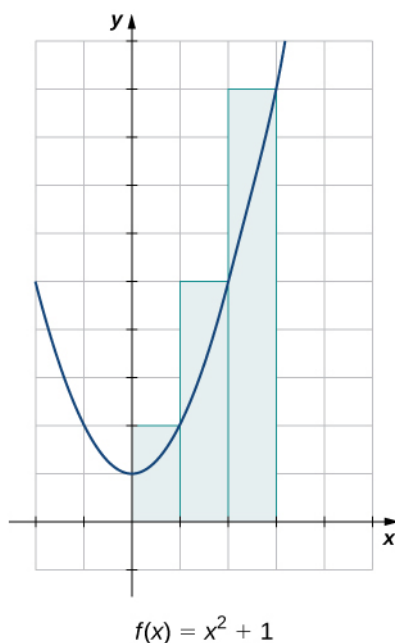

$$f(x) = x^2 + 1$$

Figure 2.10 The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

Solution

The areas of the three rectangles are 1 unit^2, 2 unit^2, and 5 unit^2. Using these rectangles, our area estimate is 8 unit^2.

2.3 Estimate the area between the x-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown here:

$$f(x) = x^2 + 1$$

Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions. Suppose, for example, that instead of determining the velocity of an object moving along a coordinate axis, we want to determine the velocity of a rock fired from a catapult at a given time, or of an airplane moving in three dimensions. We might want to graph real-value functions of two variables or determine volumes of solids of the type shown in **Figure 2.11**. These are only a few of the types of questions that can be asked and answered using **multivariable calculus**. Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables. However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.
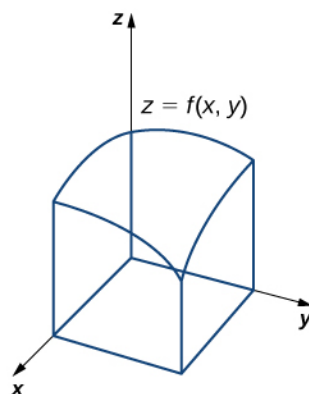

Figure 2.11 We can use multivariable calculus to find the volume between a surface defined by a function of two variables and a plane.

2.1 EXERCISES

For the following exercises, points $P(1, 2)$ and $Q(x, y)$ are on the graph of the function $f(x) = x^2 + 1$.

1. **[T]** Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x, y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

2. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x = 1$.

3. Use the value in the preceding exercise to find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

For the following exercises, points $P(1, 1)$ and $Q(x, y)$ are on the graph of the function $f(x) = x^3$.

4. **[T]** Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x, y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

5. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x = 1$.

6. Use the value in the preceding exercise to find the equation of the tangent line at point P. Graph $f(x)$ and the tangent line.

For the following exercises, points $P(4, 2)$ and $Q(x, y)$ are on the graph of the function $f(x) = \sqrt{x}$.

7. **[T]** Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x, y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
4.1	a.	e.	i.
4.01	b.	f.	j.
4.001	c.	g.	k.
4.0001	d.	h.	l.

8. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x = 4$.

9. Use the value in the preceding exercise to find the equation of the tangent line at point P.

For the following exercises, points $P(1.5, 0)$ and $Q(\phi, y)$ are on the graph of the function $f(\phi) = \cos(\pi\phi)$.

10. **[T]** Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x, y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(\phi, y)$	m_{sec}
1.4	a.	e.	i.
1.49	b.	f.	j.
1.499	c.	g.	k.
1.4999	d.	h.	l.

11. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to f at $x = 4$.

12. Use the value in the preceding exercise to find the equation of the tangent line at point P.

For the following exercises, points $P(-1, -1)$ and $Q(x, y)$ are on the graph of the function $f(x) = \frac{1}{x}$.

13. **[T]** Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x, y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q(x, y)$	m_{sec}
−1.05	a.	e.	i.
−1.01	b.	f.	j.
−1.005	c.	g.	k.
−1.001	d.	h.	l.

14. Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to f at $x = -1$.

15. Use the value in the preceding exercise to find the equation of the tangent line at point P.

For the following exercises, the position function of a ball dropped from the top of a 200-meter tall building is given by $s(t) = 200 - 4.9t^2$, where position s is measured in meters and time t is measured in seconds. Round your answer to eight significant digits.

16. **[T]** Compute the average velocity of the ball over the given time intervals.
   a. [4.99, 5]
   b. [5, 5.01]
   c. [4.999, 5]
   d. [5, 5.001]

17. Use the preceding exercise to guess the instantaneous velocity of the ball at $t = 5$ sec.

For the following exercises, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h(t) = 15t - 4.9t^2$.

18. **[T]** Compute the average velocity of the stone over the given time intervals.
   a. [1, 1.05]
   b. [1, 1.01]
   c. [1, 1.005]
   d. [1, 1.001]

19. Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

For the following exercises, consider a rocket shot into the air that then returns to Earth. The height of the rocket in meters is given by $h(t) = 600 + 78.4t - 4.9t^2$, where t is measured in seconds.

20. **[T]** Compute the average velocity of the rocket over the given time intervals.
   a. [9, 9.01]
   b. [8.99, 9]
   c. [9, 9.001]
   d. [8.999, 9]

21. Use the preceding exercise to guess the instantaneous velocity of the rocket at $t = 9$ sec.

For the following exercises, consider an athlete running a 40-m dash. The position of the athlete is given by $d(t) = \frac{t^3}{6} + 4t$, where d is the position in meters and t is the time elapsed, measured in seconds.

22. **[T]** Compute the average velocity of the runner over the given time intervals.

    a.   [1.95, 2.05]

    b.   [1.995, 2.005]

    c.   [1.9995, 2.0005]

    d.   [2, 2.00001]

23. Use the preceding exercise to guess the instantaneous velocity of the runner at $t = 2$ sec.

For the following exercises, consider the function $f(x) = |x|$.

24. Sketch the graph of f over the interval $[-1, 2]$ and shade the region above the x-axis.

25. Use the preceding exercise to find the exact value of the area between the x-axis and the graph of f over the interval $[-1, 2]$ using rectangles. For the rectangles, use the square units, and approximate both above and below the lines. Use geometry to find the exact answer.

For the following exercises, consider the function $f(x) = \sqrt{1 - x^2}$. (*Hint*: This is the upper half of a circle of radius 1 positioned at $(0, 0)$.)

26. Sketch the graph of f over the interval $[-1, 1]$.

27. Use the preceding exercise to find the exact area between the x-axis and the graph of f over the interval $[-1, 1]$ using rectangles. For the rectangles, use squares 0.4 by 0.4 units, and approximate both above and below the lines. Use geometry to find the exact answer.

For the following exercises, consider the function $f(x) = -x^2 + 1$.

28. Sketch the graph of f over the interval $[-1, 1]$.

29. Approximate the area of the region between the x-axis and the graph of f over the interval $[-1, 1]$.